PRINCIPLES OF ANALYSIS
LECTURE 23 - INTEGRATION PROPERTIES

PAUL L. BAILEY

Proposition 1. *Let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be integrable on* $[a, b]$*. Then* $f + g$ *is integrable on* $[a, b]$*, and*

$$\int_a^b (f + g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx.$$

Proof. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Then

$$m_f(P, i) + m_g(P, i) \leq m_{f+g}(P, i) \leq M_{f+g}(P, i) \leq M_f(P, i) + M_g(P, i)$$

for every $i = 1, \dots, n$. Therefore

$$L_f(P) + L_g(P) \leq L_{f+g}(P) \leq U_{f+g}(P) \leq U_f(P) + U_g(P). \tag{1}$$

Next we would like to say something like this. Since this is true for every partition P, we have

$$\underline{\int_a^b} f\, dx + \underline{\int_a^b} g\, dx \leq \underline{\int_a^b} (f + g)\, dx \leq \overline{\int_a^b} (f + g)\, dx \leq \overline{\int_a^b} f\, dx + \overline{\int_a^b} g\, dx.$$

However, this path actually is more difficult to justify than it first appears. It is easier to proceed as follows:

Inequality (1) implies that

$$(U_f(P) - L_f(P)) + (U_g(P) - L_g(P)) \geq U_{f+g}(P) - L_{f+g}(P) \geq 0.$$

Let $\epsilon > 0$; then there exists a partition P_1 such that $U_f(P_1) - L_f(P_1) < \frac{\epsilon}{2}$, and there exists a partition P_2 such that $U_g(P_2) - L_g(P_2) < \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2$; then

$$U_{f+g}(P) - L_{f+g}(P) \leq (U_f(P) - L_f(P)) + (U_g(P) - L_g(P)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f + g$ is integrable. Moreover, $\int_a^b f\, dx - \frac{\epsilon}{2} < L_f(P)$, $\int_a^b g\, dx - \frac{\epsilon}{2} < L_g(P)$, $\int_a^b f\, dx - \frac{\epsilon}{2} > U_f(P)$, and $\int_a^b g\, dx - \frac{\epsilon}{2} > U_f(P)$; therefore

$$\int_a^b f\, dx + \int_a^b g\, dx - \epsilon \leq \int_a^b (f + g)\, dx \leq \int_a^b f\, dx + \int_a^b g\, dx + \epsilon.$$

Since this is true for every ϵ, we must have

$$\int_a^b (f + g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx.$$

□

Date: December 1, 2003.

Proposition 2. *Let* $f : [a, b] \to \mathbb{R}$ *be integrable and let* $c \in \mathbb{R}$ *Then* $cf : [a, b] \to \mathbb{R}$ *is integrable, and*

$$\int_a^b cf\,dx = c\int_a^b f\,dx.$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Then

$$cm_f(P, i) = m_{cf}(P, i) \leq M_{cf}(P, i) = cM_f(P, i),$$

for $i = 1, \ldots, n$. Thus

$$cL_f(P) = L_{cf}(P) \leq U_{cf}(P) = cU_f(P).$$

Assume $c \geq 0$. Then for any bounded set X, we have $c \sup X = \sup\{cx \mid x \in X\}$. This gives

$$c\underline{\int}_a^b f\,dx = \underline{\int}_a^b cf\,dx \leq \overline{\int} cf\,dx = c\overline{\int} f\,dx;$$

since f is integrable, the result follows in this case.

The case of $c = -1$ we leave as an exercise. It follows from these facts:

(a) $-m_f(P, i) = M_{-f}(P, i)$;

(b) for any hounded set X, we have $-\sup X = \inf\{-x \mid x \in X\}$.

□

Proposition 3. *Let* $f : [a, b] \to \mathbb{R}$*, and let* $c \in [a, b]$*. Then* f *is integrable on* $[a, b]$ *if and only if* f *is integrable on* $[a, c]$ *and on* $[c, b]$*, in which case we have*

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx.$$

Proof. Suppose that f is integrable on $[a, c]$ and on $[c, b]$, and let $\epsilon > 0$. Then there exist partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U_f(P_1) - \frac{\epsilon}{4} < \int_a^c f\,dx < L_f(P_1) + \frac{\epsilon}{4},$$

and

$$U_f(P_2) - \frac{\epsilon}{4} < \int_c^b f\,dx < L_f(P_2) + \frac{\epsilon}{4}.$$

Let $P = P_1 \cup P_2$; this is a partition of $[a, b]$. Adding these inequalities yields

$$U_f(P) - \frac{\epsilon}{2} < \int_a^c f\,dx + \int_c^b f\,dx < L_f(P) + \frac{\epsilon}{2}.$$

Therefore $U_f(P) - L_f(P) < \epsilon$, so f is integrable on $[a, b]$, and the above inequality implies that

$$\int_a^b f\,dx - \frac{\epsilon}{2} < \int_a^c f\,dx + \int_c^b f\,dx < \int_a^b f\,dx + \frac{\epsilon}{2}.$$

Since this is true for every ϵ, we have

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx.$$

Suppose that f is integrable on $[a, b]$, and let $\epsilon > 0$. Then there exists a partition $P = \{x_0, \dots, x_n\}$ such that $U_f(P) - L_f(P) < \epsilon$, and we may assume that $c \in P$, so that $c = x_k$ for some k. Then $P_1 = \{x_0, \dots, x_k\}$ is a partition of $[a, c]$, and $P_2 = \{x_k, \dots, x_n\}$ is a partition of $[c, b]$.

Clearly $U_f(P) = U_f(P_1) + U_f(P_2)$ and $L_f(P) = L_f(P_1) + L_f(P_2)$. Then

$$(U_f(P_1) - L_f(P_1)) + (U_f(P_2) - L_f(P_2)) < \epsilon.$$

Since each summand is positive, each is less than epsilon, which proves the f is integrable on $[a, c]$ and on $[c, b]$. □

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu